Multiple *"parallel"* D-branes seen as leaves of foliations and Duminy's theorem

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Abstract

We try to give a *qualitative* description of the *Godbillon-Vey class* and its relation on the one hand to the *holonomy* and on the other hand to the *topological entropy* of a foliation, using a remarkable theorem proved recently by G. Duminy (which still remains unpublished), relating these three notions in the case of *codim-1 foliations*. Moreover we shall investigate its possible consequences on string theory. In particular we shall present a conceptual argument according to which the *curvature of the B-field* (rank two anti-symmetric tensor) of open strings might be related to the Godbillon-Vey class using a *suitable generalisation* of *"Non-Abelian Geometry"* which has just appeared in physics literature. Our starting point again is the Connes-Douglas-Schwarz article on compactifications of matrix models to noncommutative tori.

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0.1 Introduction and motivation

In a number of papers for some time now (see [1], [2]), we have been trying to understand some of the underlying *topology* of M-Theory. Our approach is along the lines of [9], namely *noncommutative geometry.* In the most recent article [1] we tried to compute the transition amplitudes between some noncommutative vacua of M-Theory using the Godbillon-Vey class as a Lagrangian density in the particular case of codim-1 foliations of the 3-torus T^3. The real motivation for that (from the point of view of physics) was an attempt towards M-*Field* Theory, bearing in mind that in ordinary Yang-Mills theory, instantons are "semiclassical" particles and can be thought of as the first level of approximation in the perturbation series expansion of the underlying quantum field theory of non-abelian gauge fields. Apart from being really topological objects, instantons in fact provide the link between quantum mechanics and classical vacua (solutions of the Yang-Mills equations). On the other hand, *Matrix models* are in fact *quantum mechanics* and not really field theory. Hence we were trying to go one step further from the BFSS or IKKT matrix models for M-Theory used in the Connes-Douglas-Schwarz paper where a relation between Matrix models and D=11 supergravity was also exhibited. As it was clearly indicated there, an equivalent picture of some noncommutative background could be obtained by assuming "turning on" a *constant* 3-form field, the well-known 3-form field C of 11-dim supergravity. Anyway, 11-dim supergravity is (one of) the *limit(s)* of M-Theory and not M-Theory itself. So our aim was to try to pursue this Connes-Douglas-Schwarz idea further, considering a 3-form field which was *not constant*, assuming of course that the constant case is a special situation. Thus the corresponding D=11 supergravity picture would be that *not all of supersymmetries* are preserved. (Just recall from [9] that it was maximal supersymmetry certain rather mild restrictions on the differential forms used, this lead to *foliations.* For mathematical convenience we used S-duality and prefered to work with D5-branes instead of membranes.

In this article, in order to make clear the motivation behind [1], we shall try to exhibit, mainly via examples, what the role of the Godbillon-Vey class (abreviated to "GV" in the sequel) in the geometry of foliated manifolds is and thus give a better physical interpretation for the path integral we tried to calculate in [1]. The surprising result is that the *Godbillon-Vey class in a*

subtle way counts the topological entropy of the foliation and then following statistical physics this can, in principle, be related to the energy, although unfortunately we cannot at this stage give any explicit formula. To do that we shall be based on a truly remarkable theorem proved recently by Duminy relating the Godbillon-Vey class of a foliation with the *holonomy* of its leaves and with the *topological entropy* of the foliation seen as a generalised dynamical system. "En route" we shall attempt to draw some analogies with Chern classes for bundles used in ordinary instantons. As it is well known, Chern classes count the "number of global twists" of a bundle which represents physically a soliton solution of the Yang-Mills equations and that is related to the energy of the solution (classical vacuum). *Hence one of the points in this piece of work is that we suggest an analogous physical interpretation for the GV-class but one has to adopt the statistical physics approach.* We shall also argue that the *"Non Abelian Geometry"* introduced recently in [5] is very closely related to the case of foliations defined by *closed* forms. The most general case that we shall consider here would be an arbitrary foliation, in which case we shall argue that the *curvature of the 2-form B-field in string theory* might be *related to the GV-class*. Then following the discussion along the lines of [5], Duminy's theorem seems to imply some kind of relation between *"entropy of monopoles"* and *noncommutativity of charges*.

0.2   Foliations

Let M be a smooth closed n-manifold. A codim-q (and hence dim-$(n-q)$) foliation F on M is given by a codim-q integrable subbundle F of the tangent bundle TM of M. Sometimes F itself is called the *tangent bundle of the foliation* and its *quotient bundle* $\nu(F) := TM/F$ is called the *normal bundle of the foliation*. This defines a *decomposition* of M into a *disjoint union* of $(n-q)$-dimensional submanifolds which are called the *leaves* of the foliation. *Thus since in general the leaves of a foliation do not intersect one can think of a foliation as defining a generalised notion of "parallelism" between its leaves.* The simplest and most trivial example of a foliation is of course Cartesian product.

The leaves of a foliation have three important properties: (i) they are connected (but may not be of course simply connected) submanifolds of M,

(ii) they are all of the *same* dimension $(n-q)$ and (iii) they are *immersed* submanifolds of M. One might immediately observe some similarities with the *total space* of a fibre bundle. In fact *the total space of a fibre bundle is the 2nd simplest example of a foliation*, but in fact still rather a trivial one, the leaves being the fibres. One has two main differences between fibrations and foliations: 1.The fibres are *embedded* submanifolds whereas the leaves need only be *immersed*; that gives a notion of "parallelism" for leaves in a foliation which is far more general than the situation for fibres in a bundle 2.All fibres are usually *diffeomorphic (or homeomorphic)* to some fixed "model" manifold which is called the "typical fibre". For foliations the situation is *drastically different*: *the leaves may not have the same fundamental group (!)* hence they may not be even homotopy equivalent and some of them may be compact and some others may not. These two differences will be of paramount importance in our discussion because they will give rise to what is called in foliation theory the *"holonomy"* of the foliation. That in turn is the source of *noncommutativity* on the corresponding C^*-algebra of the foliation (see [2]). It is worthwile mentioning an analogy here: we would like to think of foliations in noncommutative geometry in some sense as an analogue of symplectic manifolds and Poisson algebras. Any symplectic manifold gives a Poisson algebra structure on its corresponding commutative algebra of functions yet of course not every Poisson algebra can be thought of as coming from a symplectic manifold. One has more Poisson algebras than symplectic manifolds. The use of symplectic geometry though is important because it gives clearer pictures and one can get more insight. Thi e algebras and foliations. The former are far more general but in studying foliations one uses topology and can get more insight.

There is a local definition for a foliated manifold using the notion of *foliated chart and atlas*. A *foliated chart of codim-q* on a smooth closed n-manifold M is a pair (U, ϕ) where $U \subset M$ is open and $\phi : U \to B_\tau \times B_\pitchfork$ is a diffeomorphism and B_τ is a rectangular neighborhood in $\mathbf{R^{n-q}}$ where "τ" stands for *tangential* and $B_\pitchfork$ is a rectangular neighborhood in $\mathbf{R^q}$ where "$\pitchfork$" stands for *transverse*. The set $P_y = \phi^{-1}(B_\tau \times \{y\})$ where $y \in B_\pitchfork$ is called a *plaque* of this foliated chart. Similarly for each $x \in B_\tau$ the set $S_x = \phi^{-1}(\{x\} \times B_\pitchfork)$ is called a *transversal* of the foliated chart. Then a *foliated atlas of codim-q* is a collection of foliated charts $\{U_a, \phi_a\}_{a \in A}$ that cover M. *The plaques patched together form the leaves of the foliation*, hence the leaves

have dimension $(n-q)$. If we want to emphasise the transverse or tangential coordinates we shall write (U_a, x_a, y_a) for (U_a, ϕ_a), with $x_a = (x_a^1, ..., x_a^{n-q})$ i.e. tangential coordinates and $y_a = (y_a^1, ..., y_a^q)$, i.e. the transverse coordinates. On the overlap $U_a \cap U_b$ we denote g_{ab} the *tangential transition functions* and γ_{ab} the *transverse transition functions*. Only the *transverse* transition functions satisfy the *cocycle conditions*:

$$\gamma_{aa} = 1$$

$$\gamma_{ab} = \gamma_{ba}^{-1}$$

and

$$\gamma_{ac} = \gamma_{ab} \circ \gamma_{bc}$$

The set $\gamma = \{\gamma_{ab}\}_{a,b \in A}$ of the *transverse* transition functions is called the *holonomy cocycle* of the (regular) foliated atlas $\{U_a, \phi_a\}_{a \in A}$. ("Regular" is some technical convenient notion, one can prove that every foliated atlas has a regular refinement).

A codim-q foliated n-manifold (or a manifold carrying a foliation) is then defined to be an equivalence class of codim-q foliated atlases (after introducing an appropriate notion of equivalence called *coherence*). The proof that this definition using foliated atlases is equivalent to the one mentioned in the beginning involving integrable subbundles of the tangent bundle, is by no means trivial and can be found in for example [3] (Theorem 1.3.8 Frobenius Theorem p37). Note moreover that this definition of a cocycle is roughly the same used in [1] but there we called it a Haefliger or Γ_q-cocycle. There is a difference however, here we assume local diffeomorphisms essentially from $\mathbf{R^q}$ to itself whereas in [1] we assumed *germs* of local diffeomorphisms from $\mathbf{R^q}$ to itself. In the next section where we shall elaborate on the holonomy, we'll see that the first cocycle defines the *total holonomy pseudogroup* of the foliation whereas the second defines the *germinal holonomy groupoid* of the foliation; they are of course very closely related and essentially they "contain" the same information.

The key point to note in this definition is that in some sense one introduces a *topological "decomposition"* of M to *tangential* and *transverse* directions. Physicists cannot avoid comparing this structure with *supermanifolds*

in which case a manifold has also an *algebraic decomposition* into commuting and anticommuting directions (coordinates).

There is a generalisation of the above definition in the case where the *transverse* piece of the manifold is required to be *homeomorphic to an arbitrary metrizable topological space* (namely not just to some Euclidean space) but still keeping the *tangential part diffeomorphic to some Euclidean space (hence the leaves will be manifolds)*. In this far more general setting one talks about *foliated spaces* (i.e. not foliated manifolds) or *abstract laminations*. Moreover clearly *one could think of a supermanifold as a (rather trivial) lamination where the transverse space, corresponding to fermionic degrees of freedom, is a Euclidean space with reversed parity.*

Let us mention also that Frobenius theorem allows one to define a codim-q foliation via a non-singular decomposable q-form ω say, on M which vanishes exactly on vectors tangent to the leaves of the foliation. Integrability then implies that

$$\omega \wedge d\omega = 0$$

It is important to underline that *the leaves of foliations of arbitrary dimension on a manifold M are in fact the higher dimensional generalisations of flows of vector fields of M*. Equivalently they can be considered as the *orbits* of a (generalised) dynamical system.

0.3   Holonomy

Let us now elaborate on the notion of the *holonomy* of the foliation. Again, let M be a closed smooth n-manifold having a codim-q foliation F. What the holonomy of the foliation does is the following: if L is a leaf of the foliation and s a *path* in L, but usually we shall consider it to be a loop, one is interested in the behavior of the foliation in a neighborhood of s in M. Intuitively we may think of ourselves as "walking" along the path s keeping an eye on all the nearby leaves; as we walk we may see some of these nearby leaves "peeling away", getting out of visual range, others may suddenly come into range and approach L asymptotically (for simplicity we shall ignore the points of intersection of leaves which would lead to singularities; to deal with

singularities one has to use laminations; recall that in general the leaves are not normally allowed to intersect since they are "parallel"), others may follow along in a more or less parallel fashion or wind around L laterally etc. This behaviour when appropriately formalised is called the *holonomy* of the foliation. There are basically two ways to do that: one is by defining the *total holonomy pseudogroup* of the foliation or by defining the *holonomy groupoid* of the foliation (or what is called *germinal holonomy* in [3]). The second is due to Wilnkenkempern (see [8]) and it is what was used in [2] to define the corresponding C^*-algebra of the foliation and then *a new invariant for foliated manifolds*. One can see all the relevant details in [3]. The important thing to note here is that *the total holonomy pseudogroup or the holonomy groupoid of the foliation essentially contains all the information coming from the fundamental groups of each one of the leaves plus their configuration*, hence everything concerning the topology of the foliation. The key observation then is that an *infinitesimal version of the germinal holonomy* is actual odim-1 case where things are more straightforward. So *the GV-class essentially contains information about the holonomy of the foliation which is in fact responsible for the noncommutativity of the corresponding C^*-algebra! Yet, as Duminy's theorem states, this is done in a rather complicated way which is yet far from being clearly understood.*

In order to understand the notion of the holonomy of a foliation, it is better to give an example; that will involve the famous Reeb foliation (codim-1 case) of the 3-torus (for a formal discussion see [3] p15 and p45). Take the boundary leaf L of the Reeb foliated solid torus. L itself topologically is a 2-torus. Let $s(t)$ be a longitudinal loop on L of period $a > 0$ and based at $p = s(0)$. Imagine yourself walking along s. With each complete circuit of s, the walker will see the nearby leaves spiral in closer. (The picture for one to have in mind is of "a snake trying to bite its own tail" and hence swallowing itself but that happens repeatedly). Our observer could quantify this data by carrying a rod J_t, always *perpendicular* to the home leaf L and having one endpoint at $s(t)$ (in the general case of a codim-q foliation this will be a *transversal*; here because we are in codim-1 case it is just an interval). After one circuit $J_0 = J_a$ and the points of intersection of this rod with nearby leaves will all have moved closer to the endpoint $s(0) = s(a) = p$. This is called "first return map" in dynamical systems and can be viewed as a diffeomorphism $h_s : J_0 \to I$ onto a subinterval $I \subset J_0$ which also has p as

an end point. This contraction map to p is called the *holonomy* of the loop s.

Aside: In the string theory picture we shall present later on, essentially generalising the notion of *Non-Abelian Geometry* of [5], we shall think of D-branes as the leaves of some foliation and the string (or its worldsheet) probed by the D-branes as representing one (or two) of the dimensions of the transversals; the new point of view here is that the highly non-trivial *topology* that the leaves of a foliation might have can "absorb"—or "capture"—the *algebraic noncommutativity* and at the same time give clear intuition on how one might *topologically increase* this amount of noncommutativity; this corresponds exactly to the notion of *topological entropy!*

In order to give a clearer intuitive picture, let us recall another special example (of arbitrary codimension): perhaps the simplest non-trivial example of a foliation of arbitrary codimension is called a *foliated bundle* and it is in fact a *flat* principal G-bundle with total space M, structure Lie group G which the fibres are homeomorphic (or diffeomorphic) to and base space N which is *not simply-connected*. In this case one has *two* structures on the *total space*: the fibration (with fibre G) and the foliation induced by the flat connection where the leaves are *covering spaces* $\tilde{N}$ of the base space N. This case was studied in great detail in [2] where we actually proved that the C^*-algebras associated to these two structures on the total space were $C(N) \otimes K$ for the fibration and $C(N) \rtimes \pi_1(N)$ for the foliation, where K is the elementary C^*-algebra of compact operators acting as smoothing kernels along the fibres and $\pi_1(N)$ is the fundamental group of the base space N. Both these C^*-algebras are noncommutative but the first is Morita equivalent to just $C(N)$ which of course is commutative whereas the second is *not* even Morita equivalent to a commutative one. The lesson we would like to extract from this example is the following: *the true origin of the noncommutativity of the corresponding C^*-algebra of a foliated bundle lies in the fundamental groups of its leaves(!)* which are *also responsible for the foliation not having trivial total holonomy pseudogroup or trivial holonomy groupoid.* We should keep that in mind for later discussions. (Note: in this example only the fundamental groups play a role in noncommutativity; in full generality "parallelism" will also enter the scene).

Let us also mention that in the above example of foliated bundles, the to-

tal holonomy pseudogroup of the foliation is calculated to be the following: as it is well-known a (gauge equivalence class of a) flat connection corresponds (this is a 1:1 correspondence in fact) to a (conjugacy class of an) irreducible representation a of the fundamental group of the base space to the structure group of the bundle, namely $a : \pi_1(N) \to G$. In this case the total holonomy pseudogroup of the foliated bundle is actually a group, the image of $\pi_1(N)$ into G by a. Note moreover that in the foliated bundle case the fibration and the foliation are transverse to each other.

An arbitrary foliation of codim-q in general does not admit a global cross section, which by definition is a transversal intersecting all leaves; it certainly admits local ones though. (In the previous example of a foliated bundle all the *fibres* are *global cross sections* of the foliation.) Then the above construction can be done only *locally* and not globally; that is the reason why in general we end up with pseudogroups instead of groups and the analogue of G will be $Diff^+(\mathbf{R}^q)$. Going back to our codim-1 example of the Reeb foliation of the solid torus, the holonomy h_s of the path $s(t)$, with $t \in [0, 1]$ for a foliation of codim-q will now be a map $h_s : S_{s(0)} \to S_{s(1)}$ where $S_{s(0)}$ and $S_{s(1)}$ are *transversals* (well in fact some open neighborhoods of transversals) of the foliation (the analogues of the "rod" J_0 and I in the Reeb foliation example; in that case they are 1-dim, hence intervals). All such local diffeomorphisms h_s form a *pseudogroup* denoted Γ_U relative to the regular foliated atlas U. It is a pseudogroup and not a group because transformations are not globally defined, hence composition may not always be defined. (Note however that pseudogroups are closed under the operation of *amalgamation*). It is clear we think that *elements of* Γ_U can be thought of as *transformations between transverse q-manifolds* defined by *"sliding alond leaves"*. In fact this is exactly the holonomy cocycle of the foliation. Notice the dependence on the foliated atlas; in fact this is not too bad, essentially all regular atlases of a foliation contain "the same information". For a detailed discussion see [4].

The above discusion can be done using the *germs* $\hat{h}_s$ instead of local diffeomorphisms h_s. Recall that the germ $\hat{h}_s$ is the equivalence class of all local diffeomorphisms agreeing with h_s in small neighborhoods. Let $S :=$ $\coprod_{a \in A} S_a$ be the disjoint union of all the transversals of our codim-q foliation. Then if y, z are in S and they lie on the same leaf we define $G_y^z := \{h_s | s$ a

path in L from y to z}. If they do not lie on a common leaf we set $G_y^z := 0$. One has a *natural composition* $G_y^z \times G_x^y \to G_x^z$ defined by $(\hat{h}, \hat{f}) \to h \,\hat{\circ}\, f$.(This will be important later in our string theory discussion because through these one can see the splitting or joining of open strings which give rise to the noncommutativity of the open string field functionals). Moreover for each $x \in S$ G_x^x contains the identity. Thus $G_S := \coprod_{y,z \in S} G_y^z$ is a *groupoid* (a small category with inverses) and it is called the *holonomy groupoid* or the *germinal holonomy* of the foliation. G_S is in fact an element of the set $H^1(M; \Gamma_q)$ according to the terminology of [1] and by dividing this by a homotopy equivalence relation one gets the topological category denoted $\Gamma_q(M)$ in [1]. Then $\Gamma_q(-)$ is a homotopy invariant functor and it was used to define K-Theory No2 in [1], where M is our smooth closed n-manifold carrying the codim-q foliations. If $y \in S$ and L is the leaf of our foliation F through y, then G_y^y is actually a *group* and it is called the *holonomy group* of the leaf L at y and it will be denoted $H_y(L)$. If s is a loop on L based at $y \in L \cap S$, then $\hat{h}_s \in H_y(L)$. This defines a surjective map which is in fact a group homomorphism (for the proof see [3] p60)

$$\hat{h} : \pi(L, y) \to H_y(L)$$

This is called the *germinal holonomy of the leaf L*.

Let us close this section by remarking that another way to think of foliations is as spaces carrying a G-action but now G is not a Lie group as it is the case in Yang-Mills theory but it is actually a pseudogroup or a groupoid (the ones attached to the foliation using its holonomy). So they can be thought of as generalising principal G-bundles for G being a groupoid. This approach has been used in [7]. Finally another important point to remember is that foliations from the mathematical point of view are not the most general structures arising from bundles but they are the most general structures known up to now that one can still do K-Theory, define characteristic classes (using the Gelfand-Fuchs cohomology) and eventually have index theorems for leafwise elliptic operators.

Aside: It was argued in [10] that Morita equivalent algebras give rise to the same physics. This combined with the fact that Morita equivalent algebras have the same K-Theory and cyclic cohomology actually suggest

that from the noncommutative topology point of view an algebra should not only be noncommutative but more than that it should not even be Morita equivalent to a commutative one in order to be interesting from the noncommutative topology point of view. Also recall that from Serre-Swan theorem the K-Theory of commutative C^*-algebras coincides with Atiyah's original K-Theory for spaces and moreover cyclic cohomology for commutative algebras (almost) coincides with the de Rham cohomology of the underlying space.

Summarizing then, in this section we tried to exhibit mainly via examples, that the noncommutativity in foliations arises from the fact that the leaves of a foliation may have different fundamental groups and the notion of the holonomy of the foliation essentially captures the information of the fundamental groups of all of its leaves. Moreover one has a more general notion of parallelism since the leaves are only immersed submanifolds and not (as is the case for fibrations) embedded.

0.4   The Godbillon-Vey class

The purpose of this section is to try to give a flavour of "what the Godbillon-Vey class does" for foliations geometricaly. We know for instance that *the Chern classes for bundles essentially count the number of twists* that the bundle may have and thus give a non-trivial topology. This fact is used in Yang-Mills equations to "tell one classical solution from another", where the Chern classes are in fact the topological charges for the soliton solutions. Yet this topological charge is directly related to the *energy*, hence between topologically distinct vacua there is an energy barrier, roughly analogous to the *difference* of the topological charges (in fact 2nd Chern classes) of the two vacua (bundles) involved; using the quantum mechanical property of barrier penetration then one can give a physical interpretation of instantons. In [1] we tried to compute some *transition amplitudes* between *noncommutative vacua* using the GV-class as a Lagrangian density. We knew that these noncommutative vacua can be seen as *topologically distinct foliations* of the underlying manifold, hence the *GV-class* can indeed be used as a *topological charge* to distinguish one solution from another. Its physical interpretation though was not clear. Here we try to understand the GV-class better in ge-

ometric terms and then discuss its possible physical interpetation. We think that it can be related to the *entropy* of the foliation initially and then using statistical physics, to the *energy*. This interpretation seems rather surprising and we think, interesting.

So the story goes as follows: for convenience, let us restrict ourselves to the *codim-1* case where things are clearer. A foliation F on a smooth n-manifold M as in the previous section can be defined by a non-singular 1-form ω vanishing exactly at vectors tangent to the leaves. Integrability of the corresponding $(n-1)$-plane bundle F of TM implies that $\omega \wedge d\omega = 0$ or equivalently $d\omega = \omega \wedge \eta$ where η is another 1-form. The 3-form $\eta \wedge d\eta$ is closed hence determines a de Rham cohomology class called the *Godbillon-Vey* class of F. Although ω is only determined by F up to multiplication by nowhere vanishing functions and η is determined by ω only up to adition of a d-exact form, actually the Godbillon-Vey class depends only on the foliation F. The Godbillon-Vey class can also be defined for foliations of codim grater than 1 as we explained in [1] and equivalently η can be thought of as a *basic* connection on the normal bundle.

Now here is the **key** *observation:* for a codim-q foliation F, the *Jacobian* J of the *germinal holonomy homomorphism h of a leaf L* of F defines a group homomorphism $Jh : \pi_1(L) \to GL(q; \mathbf{R})$ called *infinitesimal germinal holonomy homomorphism* of the leaf L. For *codim-1 case* one gets simply $\mathbf{R}^+$ as the range of the group homomorphism and it is customary in this case to assume composition with *log* function and eventually get a map from the fundamental group of the leaf to $\mathbf{R}$. On the other hand, in algebraic topology there is a standard identification of the first cohomology group $H^1(L; \mathbf{R})$ with the set of group homomorphisms $\psi : \pi_1(L) \to \mathbf{R}$ so one can view the composition $log \circ Jh$ as an element of $H^1(L; \mathbf{R})$. In particular when $H^1(L; \mathbf{R}) = 0$, the infinitesimal germinal holonomy of the leaf is trivial. The 1-form η used in the definition of the GV-class restricted to L is *closed* and its class in $H^1(L; \mathbf{R})$ is exactly the infinitesimal germinal holonomy class $log \circ Jh$! This suggests that $GV(F) = [\eta \wedge d\eta] \in H^3(M; \mathbf{R})$ *should be tied with the holonomy of the foliation!* (**Aside:** *This observation largely explains the similarities observed in [1] between codim-1 foliations of a 3-manifold defined by closed 1-forms and Abelian Chern-Simons theory* on-shell; "on-shell" in Chern-Simons theory means "flat" and hence in the Abelian case just "closed"). At this point

we meet Duminy's theorem saying that this is indeed true but the relation between the GV-class and holonomy is not very straightforward.

Note: In general the infinitesimal germinal holonomy of a leaf L of a codim-q foliation F on M is an element of $Hom(\pi_1(L); GL(q; \mathbf{R}))$. For codim-1 case then triviality of infinitesimal holonomy is equivalent to vanishing of just $H^1(L; \mathbf{R})$. This key observation is partly responsible for very special things occuring in codim-1 case, one of the most impressive ones being the famous Thurston stability theorem (which improves Reeb stability).

Before stating and explaining Duminy's result let us recall some facts about the *delicate* issue of the *invariance* of the GV-class. Recall that by definition a foliation is an *integrable* subbundle of the tangent bundle. Hence roughly one could define *two notions of homotopy* equivalence between foliations: the first assumes integrability in the intermediate steps whereas the second neglects it. To be more precise, there are actually *three* useful relations between foliations (starting from the most general and going to the most narrow these are): homotopy, concordance and integrable homotopy.

The precise definitions can be found in [3]. Here we shall try to give an intuitive picture. We shall say that two foliations F and F' of our manifold M are *integrable homotopic* if one can deform continously one to the other *through intermediate foliations* F_t where $t \in [0,1]$ with $F_0 = F$ and $F_1 = F'$. There is a variant of this definition using the notion of *isotopy* but for compact M they coincide. If we divide the set $H^1(M; \Gamma_q)$ by integrable homotopy equivalence relation, we end up with the topological category denoted $Fol_q(M)$ in [6]. This then can in principle be used to define another K-Theory using the Quillen-Segal construction as described in [1].

The second definition is the following: we shall say that two foliations F and F' of our manifold M are *concordant* if one can deform continously one to the other through *Haefliger structures* F_t where $t \in [0,1]$ with $F_0 = F$ and $F_1 = F'$. A Haefliger structure is a mild generalisation of a foliation. We shall call the foliations simply *homotopic* if the deformation is performed through arbitrary (i.e. not necessarily integrable) subbundles of the tangent bundle. *This* homotopy takes us from $H^1(M; \Gamma_q)$ to $\Gamma_q(M)$ according to the terminology in [1].

Now it is a surprising (perhaps) fact that although the GV-class is a de Rham cohomology class (and de Rham cohomology is homotopy invariant), it is not homotopy invariant according to the notion of homotopy defined just above for foliations. The GV-class is only *integrable homotopy* invariant. In the particular case of *codim-1 foliations* it is *concordance* invariant. If our manifold M is a *compact 3-manifold*, then the *GV-invariant* (i.e. the real number obtained from evaluation of GV-class against the fundamental 3-homology class $[M]$ of M), is in fact *cobordism* invariant (more general than concordance). For the proofs the interested reader may see [3] (Chapter 3).

Moreover the GV-class may vary continously and non-trivially; in fact Thurston has proved in [11] that for the 3-sphere the GV-invariant can take any non-negative real value! (again for a simplified version of this proof see [3]). This result is probably not very encouraging for the invariant we tried to define in [1]; the point here though is that in [1] we restricted ourselves to *taut* codim-1 foliations (for reasons coming both from physics and from mathematics) and S^3 has *no taut codim-1 foliations* essentially because all codim-1 foliations of the 3-sphere have a Reeb component. We would like to note that foliations in general are rather "too flexible" structures and one usually wants to make them more "rigid" and one way to do that is to restrict to "taut" ones as described in [1]. Just recall that on a closed oriented connected smooth 3-manifold M, a codim-1 foliation is called (geometrically) taut if M admits a Riemannian metric for which all leaves are minimal surfaces or equivalently if there exists a closed transversal (that cannot be anything else than S^1) which intersects all leaves. There is yet another characterisation of (geometrically) taut foliations due to Rummler in this case using forms, namely for each taut codim-1 foliation there exists a unique closed 2-form which is transverse to the foliation (namely it is non-singular when restricted to the leaves of the foliation). There is an analogous statement which goes under the name of Sullivan's theorem for codim > 1 taut foliations which we shall mention in the Appendix.

One last definition before we state Duminy's theorem: a leaf L of a codim-1 foliation F is called *resilient* if there exists a transverse arc $J = [x, y)$ where $x \in L$ and a loop s on L based on x such that $h_s : [x, y) \to [x, y)$ is a contraction to x and the intersection of L and (x, y) is non-empty. (Note that

in the definition above the arc J is *transverse* to the foliation). Intuitively a resilient leaf is one that "captures itself by a holonomy contraction". The terminology comes from the French word *"ressort"* which means *"spring-like"*.

Now we shall state **Duminy's Theorem:**(see [12])

For a codim-1 foliation F on a closed smooth n-manifold M one has that $GV(F) = 0$ unless F has some (at least one) resilient leaves.

The proof of this theorem is still unpublished. A discussion about the proof has been exhibited in [3] using the theory of *levels (or "architecture") of foliations*. The authors mentioned in [3] that the full proof of Duminy's theorem **will appear** in the forthcoming Vol II of their book. We give a brief outline of the proof in the Appendix. The point here is that although in the previous section we mentioned the observation that the GV-class is related to the infinitesimal holonomy of the leaves of the foliation, now Duminy's theorem makes a far more delicate statement saying that it is in fact *only the resilient leaves* that *contribute to the GV-class* of the foliation!

The notion of resilient leaves can be generalised for foliations with codim grater than 1 although we do not know what their relation with the GV-class would be in these cases. Turning to our string theory picture that seems to suggest that in order to have a D-brane behaving like a resilient leaf there must be a string having one of its endpoints on that D-brane (that would correspond to the transverse path $[x, y)$ of the definition above) and which intersects this D-brane again at a point other than its second endpoint. Moreover there must be a loop s on the D-brane based on the string endpoint such that the D-brane captures itself by the holonomy contraction h_s induced by the loop s. (It is not perhaps very easy to visualise this).

Let us treat some special cases as examples: The GV-class *vanishes* for a special class of codim-1 foliations, those defined by *closed* 1-forms. (**Remark:** this is true for foliations defined by *constant* differential forms which is the case considered in the Connes-Douglas-Schwarz article). We suspect that it is probably also zero in general for *fibrations* although we have not been able to prove this. We mentioned that a codim-1 foliation F on a closed smooth n-manifold M can be defined by a non-singular 1-form ω on M. In-

tegrability implies $\omega \wedge d\omega = 0$. In particular, this is automatically satisfied for *closed* 1-forms. Now codim-1 foliations defined by closed 1-forms have a very special property: they are *homeomorphic* to foliations with *no holonomy*, see for example [3] section 9.3. It is rather obvious, as was also pointed out in [1] when comparing codim-1 foliations on a 3-manifold and Abelian Chern-Simons theory, that *a codim-1 foliation defined by a closed 1-form* (i.e. Abelian Chern-Simons on-shell) *has vanishing GV-class*. Now if the codim-1 foliation has trivial holonomy then it is homeomorphic to a foliation defined by a closed 1-form but homeomorphisms do not respect smooth forms and hence we cannot conclude that the GV-class vanishes. This is *true however* being a corrolary to Duminy's theorem since such foliations *cannot have resilient leaves* (see [13]).

The relation that all this has with bundles is the following: a codim-1 foliation F defined by a closed 1-form ω on M has automatically a *transverse holonomy invariant measure* μ, where $\omega := d\mu$ is closed. Denote by $P(\mu)$ the group of *periods* of the measure μ which by definition is the image of the group homomorphism $[\mu] : \pi_1(M) \to \mathbf{R}$ (and hence itself can be seen also as an element of $H^1(M; \mathbf{R})$) defined by

$$[\mu]([\sigma]) := \int_\sigma d\mu$$

One can prove that as cohomology classes $[\omega] = [\mu]$ in $H^1(M; \mathbf{R})$ (see [3] Proposition 9.3.4 p219).

The *rank* $p(\mu)$ of μ is then by definition the rank of the group $P(\mu)$. Then one has

Proposition 1:
F is a fibration if and only if $p(\mu) = 1$.

(For the proof see [3] p219). Hence one can get a characterisation of M being a codim-1 foliated manifold or the total space of a fibration. The later is a special case of a codim-1 foliation defined by a closed 1-form. Moreover a theorem of Tischler states that codim-1 foliations defined by closed non-singular 1-forms can be smoothly and uniformly "well approximated" by fibrations over S^1 (see [3] p221). Note moreover that if a closed smooth

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n-manifold admits a codim-1 foliation defined by a closed 1-form that would imply that its first cohomology group does not vanish.

The lesson from all this discussion is that at least in the codim-1 case, foliations defined by *closed* 1-forms are *"the next closest thing to a fibre bundle"*, namely they are homeomorphic to foliations with no holonomy and have vanishing GV-class. In dimension 3 we know that the classification of manifolds is not category dependent hence there is no difference between homeomorphism and diffeomorphism and consequently in dim 3 a foliation with no holonomy is the same as a foliation defined by a closed 1-form. Recall that in the previous section we mentioned that *the GV-class measures in a subtle way the noncommutativity of the foliation* (whose origin is the holonomy); here *we saw an indication of the validity of that statement* (at least for the codim-1 case), coming up in a complicated way: in [2] we saw that the corresponding C^*-algebra of an *arbitrary* fibre bundle, although it is noncommutative, it is in fact *Morita equivalent to a commutative one* (hence, seen as a foliation, it has no holonomy); on the other hand, codim-1 foliations defined by *closed 1-forms are very closely related (in fact they are homeomorphic) to codim-1 foliations having no holonomy*, a *special case* of which are *fibrations over S^1* and finally *in all these cases the GV-class vanishes* indeed (as a consequence of Duminy's theorem).

0.5   Topological Entropy

Our discussion above concerned primarily codim-1 foliations. This will still be mainly the case in this section also. Let us start with the following remark: in higher codimensions things become much more complicated and the study of foliations involves *statistical techniques* borrowed from *ergodic theory*. It is in this framework that the important notion of *topological entropy* arose historically for the first time. The fundamental result in this section is the remarkable consequence of Duminy's theorem (recall that this holds for codim-1 foliations) that: *"non-vanishing GV-class implies positive entropy"* for the foliation!

To begin with we shall define the notion of *entropy of maps* and then

we shall generalise it for foliations using as intermediate steps the entropy of transformation groups and pseudogroups.

In general, *entropy measures the rate of creation of information.* Roughly, if the states of a system are described by iteration of a map, states that may be *indistinguishable* at some initial time may diverge into clearly *different* states as time passes. Entropy measures the *rate* of creation of states. In the mathematical language it measures the *rate of divergence of orbits of a map.*

There are *two* concepts of entropy for maps: *topological* and *measure-theoretic.* We shal consider mainly the first here due to Ghys, Langevin and Walczak (see [14]). The idea is to *define the entropy of the foliation as the entropy of its corresponding holonomy groupoid (or equivalently its holonomy pseudogroup)* in close analogy to the definition of the entropy of a map. The topological entropy of a foliation is closely related to two other notions: the *growth type* of its leaves (assuming that the leaves have Riemannian metrics we study the "evolution" of their volume) and the existence of *transverse holonomy invariant probabilistic measures.* In fact zero entropy implies the existence of a transverse invariant measure. We shall not discuss further these notions here.

Now suppose f is a map of a compact manifold into itself. To measure the number of orbits one takes an empirical approach, not distinguishing e-close points for a given $e > 0$. If x and y are two indistinguishable points, then their orbits $\{f^k(x)\}_{k=1}^{\infty}$ and $\{f^k(y)\}_{k=1}^{\infty}$ will be distinguishable provided that for some k, the points $f^k(x)$ and $f^k(y)$ are at distance grater than e. Then one counts the number of distinguishable orbit segments of length n for fixed magnitude e and looks at the growth rate of this function of n. Finally one improves the resolution arbitrarily well by letting $e \to 0$. The value obtained is called *the entropy of f* and it measures the asymptotic growth rate of the number of orbits of finite length as the length goes to infinity.

More formally, let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map and e a positive real number. We say that a set $E \subset X$ is (n, e)-*separated by f* if whenever x, y are two points of E, then $d(f^k(x), f^k(y)) \geq e$ for some $0 \geq k \geq n$. The maximum number of pairwise

e-distinguishable orbits of length n is

$$S(f, n, e) := sup\{card(E)|Eis(n, e) - separated by f\}$$

This number is finite because X is compact. Its growth rate h is by definition

$$h(f, e) := lim_{n \to \infty} sup\frac{1}{n} log S(f, n, e)$$

This number is possibly infinite and nondecreasing as $e \to 0$.
Definition: The *entropy of f* is the nonnegative or ∞ number

$$h(f) := lim_{e \to 0} h(f, e)$$

Example 1: If f is an *isometry* then its entropy is zero.

Example 2: If A is an $n \times n$ matrix with integer entries, then A defines an endomorphism of the n-torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$. If $l_1, ..., l_n$ are the eigenvalues of A, then $h(f) = \sum_{|l_i|>1} log|l_i|$.

The next step is to define the *entropy of transformation groups.* Let G be a group of homeomorphisms of the metric space (X, d) with finite generating set G'. We assume that this generating set is *symmetric*, namely G' contains the identity transformation and is invariant under the operation of passing to the inverse. We set G_n equal to the set of elements of G that can be written as *words of length $\leq n$ in elements of G'*. We say that points $x, y \in X$ are (n, e)-*separated by G'* if there exists $g \in G_n$ such that $d(g(x), g(y)) \geq e$. A subset E of X is (n, e)-*separated by G'* if each pair of distinct points in E is so separated. The supremum of the cardinalities of such sets is denoted $S(G, G', n, e)$ and we define the *entropy h* as

$$h(G, G') := lim_{e \to 0} lim_{n \to \infty} sup\frac{1}{n} log S(G, G', n, e)$$

Then we can define the *entropy of pseudogroups.* We fix a (possibly non-compact) metric space (X, d) together with a pseudogroup Γ of local homeomorphisms of X that admits a finite symmetric generating subset Γ' containig the identity. For each positive integer n, let Γ_n denote the collection of elements of Γ that can be obtained by composition of *at most n elements of*

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Γ' and let Γ_0 consist of the identity of X. Now let $e > 0$. Points $x, y \in X$ are said to be (n, e)-separated by Γ' if there exists $f \in \Gamma_n$ whose domain contains x and y and such that $d(f(x), f(y)) \geq e$. A subset E of X is defined to be (n, e)-separated if every pair of distinct points x, y in E are (n, e)-separated. Let then $S(n, e)$ be the supremum of the cardinalities of the (n, e)-separated subsets of X. Then the *entropy of Γ with respect to Γ'* is by definition the nonnegative or ∞ *integer*

$$h(\Gamma, \Gamma') := lim_{e \to 0} lim_{n \to \infty} sup \frac{1}{n} log S(n, e)$$

One can prove that this number is *independent* of metric d compatible with the topology of X for *regular pairs* (Γ, Γ') (for the proof see [3] p353).

One then can define *the entropy of a compact foliated manifold (F,M) with respect to a regular foliated atlas* $(U_a, \phi_a)_{a \in A}$ as the entropy of its corresponding *total holonomy pseudogroup* Γ_U (with respect to the regular foliated atlas). This is a *finite* integer for M compact. This definition can be generalised for foliated spaces.

Remark: Unfortunatelly (perhaps), the topological entropy of a foliation depends on the choice of a regular foliated atlas, hence *it is not an invariant of the foliation*. That creates problems if one tries to relate that to the energy in physics in a similar fashion that the entropy of a statistical system is related to the energy. One should define something like *an average of the topological entropy*. This situation is not unusual in topology; average values of topological invariants can be defined (e.g. *average Euler characteristic* for surfaces), see for example [16]. It is *merely the vanishing or non-vanishing* of the topological entropy which is an *invariant of the foliation(!)* and not the precise value of the topological entropy.

Let us give some examples: Cartesian products of compact manifolds considered as trivial foliations have zero topological entropy. The famous Reeb foliation of S^3 has zero topological entropy. Moreover if two foliations on the same manifold and of the same codimension are *integrable homotopic* and one of them has topological entropy zero, so has the other. Hence the condition "zero topological entropy" is an integrable homotopy invariant! (For the proof see [3] p362).

Perhaps the clearest example where one can see very clearly the role of the notion of (n, e)-separation and how it is used to define the topological entropy is the following elementary one from electromagnetism in dim-3 space (1-dim foliation): a *uniform* electric field given by the equation $\vec{E} = const$ is an example of a dynamical system with zero topological entropy, since the flow lines of the electric field are always "parallel" and they do not *diverge,* namely the "distance" between them remains the same; hence there is no creation of information, states (i.e. flow lines which are the leaves of our 1-dim foliation) which are indistinguishable remain so for ever, using the notion of (n, e)-separation. This is in sharp contrast to the case of the flow lines of the electric field created by a *point charge.* In this case the entropy is *strictly positive* since there is creation of information: because the flow lines diverge, states (i.e. flow lines) which are very close to each other at some initial time and hence they are indistinguishable, they will eventually become distinguishable because the "distance between them" will increase. In the above example the flow lines in both cases are straight lines. Note that for the topological entropy what is important is not the shape of the flow lines (they could be arbitrary curves) but whether they diverge or not (namely if the distance between them remains the same or not). So an electric field which is not constant can still have zero entropy as long as the flow lines (being arbitrary curves) do not diverge. On the other hand the point charge creates an electric field which has non-zero topological entropy although its flow lines are straight lines.

There is a notion of *geometric entropy* for a foliation using a *leafwise Riemannian metric* and in fact the main theorem proved in [14] indicates the relation between topological and geometric entropy for foliations. We shall not consider this here.

We now pass on to the main subject of this section, namely the relation between topological entropy and the GV-class:

Proposition 2:
If the compact foliated space (M, F) has a *resilient* leaf, then F has *positive* entropy.

For the proof see [3] p379.

The notion of resilient leaves can be generalised for the case of foliated spaces of arbitrary codim as follows: a loop s on a leaf L based at $x \in L$ is contracting if a holonomy transformation $h_s : D \to D$ associated to s and defined on a suitable compact transverese metric disc D centered at x, is such that $\cap_{n=1}^{\infty} h_s^n(D) = \{x\}$. The leaf L is *resilient* if it has such a contracting loop and $L \cap (intD - \{x\}) \neq \emptyset$.

Combining this with Duminy's theorem (for *codim-1 case*) we get the following:

Corollary:
If (M, F) is a compact $(C^2$-)foliated manifold of *codim-1*, then *zero entropy implies GV(F)=0*.

The *converse is not true*. A counterexample is the famous *Hirsch foliation of the 3-torus*. This is a solid torus with a wormhole drilled out that winds around twice longitudinally while winding once meridionally, having a codim-1 foliation with the leaves being *"pair of pants"* 2-manifolds (see [3] p371). The Hirsch foliation has a resilient leaf, hence positive entropy. It is however transversely affine and hence its GV-class vanishes.

0.6 M-Theory and strings probed by "parallel" D-branes

We would like to start this section by considering M-Theory first and address the following question: *how much noncommutativity do we need* for M-Theory?

Following the original Connes-Douglas-Schwarz article, the answer will probably be: *not very much*. In fact what these authors really considered was noncommutative *deformations* of commutative algebras given by the Moyal bracket for instance. In their section 4.1 discussion where they con-

sidered the equivalent picture involving (codim-1) foliations of the 2-torus, what they used was in fact the moduli space of *linear foliations* of the 2-torus. This is a *2-manifold* and the GV-class (which would have been a *3-form* for a codim-1 foliation) vanishes, so it is of no use. Yet the notion of topological entropy is useful; *in fact the topological entropy of all these linear foliations of the 2-torus is zero*. Let us define linear foliations on the 2-torus first and then explain why they have zero entropy: a constant vector field $\tilde{X} \equiv (a, b)$ on $\mathbf{R^2}$ is invariant by all translations in $\mathbf{R^2}$, hence passes to a well-defined vector field X on $T^2 = \mathbf{R^2}/\mathbf{Z^2}$. We assume $a \neq 0$. *The foliation $\tilde{F}$ on $\mathbf{R^2}$ produced by $\tilde{X}$ has as leaves the parallel lines of slope b/a.* (Just recall the usual topological picture of the 2-torus as a rectangle). This foliation is also invariant under translations and passes to the foliation F on T^2 produced by X. Since these leaves are *parallel* in the usual sense lines, there is no creation of information, hence the topological entropy is *zero*. One can then consider two cases, rational and irrational slope (in both cases the entropy is zero): If the slope is rational, these foliations are in fact fibre bundles over S^1. If the slope is irrational, then each leaf is a 1:1 immersion of $\mathbf{R}$ and is everywhere dense in T^2 (Kronecker's theorem).

This is in accordance to the equivalent 11-dim supergravity picture presented in [9] section 6, where it was argued that *these* compactifications on the noncommutative 2-torus (namely linear foliations of the 2-torus) correspond to turning on a *constant* background field C which is the 3-form potential of 11-dim supergravity. The 3-form field C is constant because we require existence of BPS states preserving *maximal* supersymmetry. (Turning on a background field C which is an arbitrary 3-form would correspond mathematically to arbitrary foliations and physically to the case where not all of supersymmetries are preserved). Now if C is constant then its field strength dC of course is zero. That is *strongly reminiscent* of *foliations defined by closed forms* and in this case one is very tempted to recall all our previous discussion concerning *codim-1 foliations*, Duminy's theorem and foliations with no holonomy. This is certainly not a proof, we just try to draw some rather striking analogies, keeping in mind that the situation for foliations of codim grater than one may be different; well, unfortunately rather few things are known for foliations of codim grater than 1.

Until fairly recently there were *two seperated aspects* of noncommutativ-

ity: *spacetime noncommutativity* and *internal space noncommutativity*. (The topological picture we shall suggest involves foliations hence one can roughly think of the first as related to the *normal bundle* of some foliation whereas the second is related to the *tangent bundle* of the foliation. Hopefully that will become clear at the end). The origins of the former are either *M-Theory*, namely the Connes-Douglas-Schwarz tradition started in [9] where the algebra of matrix valued fields in noncommutative space was thought of as the simple tensor product of constant matrix algebra and Moyal deformation) or *open string field theory itself* which started from [17]. The origins of the later spring from the non-Abelian nature of configuration of multiple D-branes. These two have remained separated in physics literature until very recently when [5] appeared. The framework in [5] is *open string field theory* and the authors present an interesting scenario in which *spacetime noncommutativity is intertwined in a non-trivial way with the non-Abelian nature of multiple D-branes* and thus they both become parts of a new underlying *unified structure* called *non-Abelian Geometry*. This was achieved by defining a *new product* between matrix valued functions using the lattice string quantum mechanics approximation; hence in this case one has a *really noncommutative situation*, not simply deformations of commutative cases. What we shall argue in the sequel is that this underlying unified structure is exactly *foliations*. But let us start by recalling some facts about open string field theory.

Open strings interact by joining and splitting, hence one gets an algebra of open string functionals using a $*$-product on the space of open paths of the target manifold. This algebra is not commutative while retaining associativity, yet it is rather too complicated to be useful. The next step then was to approximate it by considering the string as being a lattice of a finite number of points, in fact the minimal lattice approximation uses just two points, the end points of the string. Then the $*$-product becomes a deformation of the commutative product:

$$(\Psi * \Phi)(X) = exp(\frac{i}{2}\frac{\partial}{\partial X_1^\mu}\Omega^{\mu\nu}\frac{\partial}{\partial X_2^\nu})\Psi(X_1)\Phi(X_2)|^{X_1=X_2=X}$$

The parameter of noncommutativity Ω is related to the B-field and spacetime metric g via

$$\Omega = -(2\pi a')^2 g^{-1}Bg^{-1}[1 - (2\pi a')^2 Bg^{-1}Bg^{-1}]^{-1}$$

One could deal only with situations where the *B-Field is constant* and it has been thought that introducing *curvature* for B would take string theory away from its σ model realm and hence one needs a drastic conceptual advance. We think that our foliation picture suggestion also offers this conceptual advancement. This is in accordance to the well-known fact that σ models had been described as *flat principal bundles* by Polyakov in [18] a long time ago with base the source and structure group the isometries of the target space. Flat bundles are foliated bundles, namely a special case of a foliated manifold, hence we just suggest the most obvious generalisation in this sense. The non-Abelian Geometry then in [5] deals with the case where B is *locally constant*. Clearly even in this case B is closed. Our foliation generalisation can deal with any 2-form B (in the "worst" case one need only to assume that it is decomposable).

Let us now recall some facts about the non-Abelian nature of the internal space of a multiple D-branes configuration (i.e. the "second source" of noncommutativity): a single D-brane carries a $U(1)$ gauge symmetry; yet the existence of multiple D-branes (say N of them) is thought to give rise to a *non-Abelian $U(N)$-symmetry* and not just to an Abelian $(U(1))^N$-symmetry. Let us then denote by A and F_A the corresponding potential and field strength of this $U(N)$ internal gauge symmetry. It has been clear for a long time that *the Abelian (trace) part* of F_A denoted $\tilde{F}_A$ could be *incorporated into the B-field* just by some suitable redefinition of the potential A, hence it also contributes to spacetime noncommutativity which was originated by B, by replacing B with $(B - Tr\tilde{F}_A/N)$ in our expression for the noncommutativity parameter Ω above. There is actually a gauge symmetry connecting the two. The physical picture is that the strings are probed by parallel D-branes. So then the question is: what about the non-Abelian part of F_A? (or what about the curvature of B, or both?) In [5] they *only* considered cases of *locally constant* either F_A or B, or both. Then by introducing a new product they managed to intertwine these two noncommutativities. We think that the case of locally constant B corresponds to a particular class of foliations, those defined by *closed* forms.

This *non-Abelian geometry* can have a *straightforward topological generalisation* by using an *arbitrary foliation picture* where the D-branes are not *parallel* in the usual sense but they are *parallel* in the far more general sense of

possible configurations of leaves in a foliation. This much more complicated topological configuration can in fact capture both spacetime noncommutativity and the non-Abelian nature of internal symmetry. Hence in this case our discussion about the GV-class and topological entropy will become important. Moreover we think that this also provides an appropriate framework for the generalisation of the σ model picture for string theory, in which the curvature of the B-field might aquire a new meaning: it may be related to the GV-class(!) thus "counting" (or measuring) noncommutativity as well as increasing the topological entropy of the D-branes.

It is we think clear from our previous discussion on the holonomy of foliations that the string (or its 2-dim worldsheet) having its end points on the D-branes (same or different does not really matter since *composition* is defined in the holonomy groupoid of a foliation and we also have inverses, that's what being a groupoid means) but not itself lying on the D-brane, can be seen as *representing 1 (or 2) of the dimensions of the transverse manifolds*, hence the noncommutativity of the algebra product of string functionals can be "incorporated" or "absorbed" in the foliation-like behaviour of the D-branes, thus arising as some part of the holonomy of the foliation. Then the noncommutative $*$-product can be obtained from the corresponding C^*-algebra as was described in [2]. What we actually suggest here is a *topological generalisation along the lines considered in* [5] and our previous articles [1] and [2]. We do that in order to gain some more insight. In fact in this setting then the *curvature of the B-field might be suspected to have something to do with the GV-class or the topological entropy of the multiple D-branes' configuration seen as leaves of a foliation*. This picture on the one hand captures the string functional algebra noncommutativity (since the foliation has non-trivial holonomy) and at the same time gives an answer to the question raised in [5] of a conceptual forward step from the σ model picture in order to understand the role of the curvature of B. Moreover the relation between B-field and noncommutativity still remains via the relation between GV-class and holonomy (well, for codim-1 case by Duminy's theorem for the moment). This is also in accordance to the relation of B and spacetime noncommutativity parameter Ω as described above. Since the GV-class for a codim-1 foliation is a real valued 3-form, this fits well to be used as representing the effect of the curvature of the B-field, also mediately appropriate to describe the non-Abelian part of F_A, which is a Lie algebra valued 2-form. We think

that in fact one does not have to worry about F.

Before explaining why, let us try to be as specific as we can: since Duminy's theorem strictly speaking applies (from what we know up to now) to codim-1 foliations, the physical picture for one to have in mind would probably have to involve the D8-branes coming from massive type II A string theory in order to have codim-1 foliations at least locally, in our target manifold. (Unfortunately these are the D-branes for which we have the less amount of information). It is probably more convenient to forget the time dimension completely and consider a target 9-manifold instead of a 10-manifold, where locally some D8-branes (which are 8-manifolds since we forget time) behave as leaves of a codim-1 foliation and the string(s) probed by these D8-branes represent(s) the transversal 1-manifold(s).

Now here is the argument: another way to think of this foliation picture is to assume that one has N parallel D8-branes and string(s) probed by them; then consider the *limit $N \to \infty$* and *moreover N becomes continuous*. Taking this limit produces two effects: the first is that one no longer needs the lattice string approximation, a string corresponds exactly to 1 continuous transverse dimension. (We do not know however if one obtains the *whole* $*$-algebra of the open string functionals as described in [17] when subsequently one passes to the C^*-algebra associated to the foliation. Perhaps one should sum all foliations to get that). Secondly, by doing so, we end up with precisely a codim-1 foliation where the leaves (which are 8-manifolds) are the D8-branes. Without loss of generality we may think that the picture of N parallel D-branes and strings having their endpoints on them can be equivalently described by having a single "long" and very flexible string probed by N D8-branes at N distinct points, hence essentially the string becomes a lattice of N points. These two pictures are equivalent because in both cases the configuration space is the same, namely the disjoint union of the N in number D8-branes. What we suggest here is that the non-Abelian $U(N)$ internal symmetry, which would correspond to some $U(N)$-bundle, when the string becomes continuous, will become the *tangent bundle* of a foliation!

It is perhaps clearer to think that one starts with some *non-Abelian bundle* (thus we *start* by considering the *internal space noncommutativity*) over some space which one can think of as being something like the *disjoint union*

of the N in number D8-branes; then the *strings* between them which create the *spacetime noncommutativity* can be *completely absorbed* in the codim-1 foliation picture and the noncommutativity they generate *has exactly been used to increase the topological entropy of the setting*, namely the original *non-Abelian bundle* over the D8-branes has actually become a *codim-1 foliation!* That in turn can be seen as a much more complicated topology of the D8-branes while at the same time remaining "parallel". The two noncommutativities have been united into a unique structure, foliations; one could also say that *the initial two separated noncommutativities where in fact an artifact of the lattice string approximation(!)*, although the tangent bundle and normal bundle of the foliation are reminiscent of this fact. That is the meaning of our statement in the Introduction that the noncommutativity of the charges becomes entropy of the monopoles. The link among these two is the GV-class which must be somehow related to the curvature of the B-field and (for codim-1 case) Duminy's theorem and its consequences as we have already discussed.

An important point here is that N does not become simply infinite but also continuous. We think that in this case this captures the difference between the K-Theories of bundles and foliations as described in [1], although that still remains a complicated issue, namely the relation(s) between the various K-Theories listed in [1]. If we had simply taken the limit $N \to \infty$ and N was discrete, then we would have roughly just obtained the classifying space BU of unitary bundles, which is in the realm of Atiyah's ordinary K-Theory. Yet what we want is something additional which can take us further from bundles (that's taking D-branes into account and internal space noncommutativity) and which can actually make contact with the classifying space of codim-1 foliations because we have another player in the field, spacetime noncommutativity (generated by the strings)!

A remark here: integrability is essential in the definition of a foliation in order to have surface-forming vector fields, namely the leaves are indeed manifolds, each one with a well-defined tangent bundle. Now the next question is how noncommutative the whole setting may become; at least from the mathematical point of view, if we adopt the description above that string fluctuations are responsible for spacetime noncommutativity, that can be seen as an increase of the topological entropy of the leaves (D8-branes). The

D-branes themselves however are also dynamical objects and they can fluctuate too; in such a case the manifold structure that each D8-brane carries would "disintegrate". That would in turn mean that we would have lost *integrability* in our plane-field which defined the foliation. *That end then would correspond to a contact structure.* The intermediate situations can be described by *confoliations* as we described in [1].

So to summarise, the topological point of view we try to present here is as follows: roughly we start with a non-Abelian bundle over the D8-branes (which captures internal space noncommutativity) and that is a structure with zero topological entropy; then the strings having their endpoints on the D8-branes (which fluctuate) *add more* noncommutativity (as well as topological entropy) and hence our bundle actually becomes a foliation which has positive entropy; this should be measured by the GV-class which must be related somehow to the curvature of the B-field; then assuming that *the D-branes fluctuate too*, we get a contact structure which has infinite topological intropy (nowhere integrable plane field).

Closed strings then *probed by D-branes* will probably correspond to *taut* foliations. It is not neccesary that strings and D-branes have complementary dimensions with respect to the target manifold. It may be the case that the noncommutativity is restricted only to some dimensions of the target manifold; see for instance the Connes-Douglas-Schwarz article where essentially *noncommutativity was restricted to the 2-torus* which was a *factor* of the Cartesian product decomposition of the target 11-manifold!

It is straightforward to generalise the above picture: foliations of arbitrary codim-q on an n-manifold, would correspond to q-branes (instead of strings = 1-branes) probed by $D(n-q)$-branes and the physical picture will involve an $(n+1)$-manifold where time is the extra dimension. Equivalently that would correspond to having *q noncommutativity parameters* instead of just 1 (the Ω of the two-point lattice string approximation). What we cannot obviously generalise from the above discussion in higher codimensions is contact structures. These seem to be very much related to the codim-1 case (they are in fact the opposite of codim-1 foliations).

From the point of view of foliation theory there are two important issues:

do we need these D8-branes to be compact (since there is no time dimension) and are there any restrictions concerning their fundamental groups? The point we want to make here is that there is *Thurston's stability theorem* (see [19]), which greatly improved Reeb stability theorem and which enables one to obtain information about the topology of the ambient foliated manifold by looking at the topology of the leaves. In more concrete terms then the theorem states the following:

If (M, F) is a compact, connected, transversely oriented smooth n-manifold with a codim-1 foliation F and moreover if F has at least one compact leaf L with $H^1(L; \mathbf{R}) = 0$, then every leaf is homeomorphic to L and M is homeomorphic either to $L \times [0, 1]$ with the product foliation or to the total space of a fibre bundle $p : M \to S^1$ having the fibres as leaves.

The proof of this theorem uses *nonstandard analysis* although there are simplified versions (see the references in [3]).

We do not know if any restrictions should be imposed on the topology of these D8-branes. What is really interesting, apart from trying to make the relation between the GV-class and string theory more concrete by writing some explicit formula, is to try to see if all this discussion about topological entropy has anything to do with the *black hole entropy* since D-branes are thought to play an important role there.

0.7 Appendix

We shall try to exhibit a sketch for the proof of Duminy's theorem (which recall, applies to codim-1 foliations). We shall give some definitions also from the theory of *levels* (or *"architecture"*) *of foliations*. Our reference is [3].

Let (M, F) be a codim-1 foliation F on a closed smooth oriented n-manifold M. A subset X of M is called F-*saturated* if it is a union of leaves of F. It is called a *minimal* set if it is closed, nonempty, F-saturated having itself no proper subset with these properties. (Example: in a fibre bundle each leaf, the leaves in this example are just the fibres, is a minimal set).

A leaf that belongs to a minimal set of F is said to be at *level 0*. The union of all leaves of level 0 is denoted M_0 which is compact. Now we set $U_0 := M - M_0$. In this case either $U_0 = \emptyset$ or F restricted to U_0 denoted $F|_{U_0}$ has at least one minimal set. In the first case $M = M_0$. In the second, let us denote M_1 the union of M_0 and all of the minimal sets of $F|_{U_0}$. M_1 is closed in M, hence it is also compact and we let $U_1 := M - M_1$. Again either $M = M_1$ or we obtain a compact F-saturated set M_2 as the union of M_1 and all minimal sets of $F|_{U_1}$. Then inductively one gets the following

Theorem:
There is a unique filtration

$$\emptyset := M_{-1} \subset M_0 \subseteq M_1 \subseteq ... \subseteq M_k \subseteq ... \subseteq M$$

of M by compact F-saturated subsets such that:
1. $M_k - M_{k-1}$ is the union of all minimal sets of $F|_{(M-M_k)}$ for all $k \geq 0$
2. If $M_k = M_{k+1}$ for some $k \geq 0$, then $M_k = M_{k+p} = M$ for all $p \geq 0$

We define the leaves of $F|_{(M-M_k)}$ to be at *level k*. Denote $M_* := \cup_{k=0}^{\infty} M_k$ and we say that the leaves of $F|_{M_*}$ are at *finite level*. The leaves (if any) in $M_\infty := M - M_*$ are said to be at *infinite level*. M_∞ is either empty or it contains uncountably many leaves. Moreover it has no interior, hence one cannot continue finding minimal sets at infinite levels and as a subset of M it is a Borel set and Lebesgue measurable. For an F-saturated measurable set $X \subseteq M$, one can show that the GV-class $GV(F)$ can be *intergrated* over X to define a cohomology class $GV(X, F) \in H^3(M; \mathbf{R})$. This is NOT obvious since the GV-class GV(F) is a 3-form (for codim-1 foliations) and X is a measurable subset of M. One uses some aspects of Poincare duality (see [3] p193). More cencretely then assuming M to be closed, oriented smooth n-manifold, one version of Poincare duality identifies the vector spaces

$$H^q(M; \mathbf{R}) = Hom_{\mathbf{R}}(H^{n-q}(M; \mathbf{R}), \mathbf{R})$$

as follows: given $[\phi] \in H^q(M; \mathbf{R})$ represented by the q-form ϕ, define

$$[\phi] : H^{n-q}(M; \mathbf{R}) \to \mathbf{R}$$

by

$$[\phi]([\psi]) := \int_M \phi \wedge \psi$$

and then define $GV(X, F) : H^{n-3}(M; \mathbf{R}) \to \mathbf{R}$ via

$$GV(X, F) := \int_X \eta \wedge d\eta \wedge \psi$$

One can then view the GV-class as an $H^3(M; \mathbf{R})$-valued countably additive measure on the σ-algebra of Lebesgue measurable F-saturated sets. This measure satisfies $GV(M, F) = GV(F)$, namely we get the original GV-class of the foliation if we apply the construction just described to the F-saturated set M itself. This is used in Duminy's theorem to prove that $GV(F)$ is zero unless some leaf is resilient. One has the following level-decomposition sum:

$$GV(F) = GV(M_\infty, F) + \sum_{k=0}^{\infty} GV(M_k, F)$$

If no leaf is resilient, the minimal sets are either proper leaves or open without holonomy. This is used to prove that $GV(M_k, F) = 0$ for all $k \geq 0$, hence $GV(F) = GV(M_\infty, F)$ and then the last step is to prove that this class vanishes. In fact Duminy proves that $GV(M_\infty, F) = 0$ for all C^2-foliations, whether or not they have resilient leaves.

The last comment is the following: it is a delicate issue in the theory of foliations the difference between functors $\Gamma_q(-)$ and $Fol_q(-)$ from spaces to topological categories. The former comes by dividing the set of all *Haefliger q-cocycles* (which for the special case of a Haefliger structure being a foliation is essentially is the *germinal holonomy* or the *holonomy groupoid* of the foliation) $H^1(-, \Gamma_q)$ by *(ordinary) homotopy* whereas for the second one has to divide by *integrable homotopy*. The GV-class *in general* (there are some differences for codim-1 foliations and for 3-manifolds as we have already explained) is integrable homotopy invariant hence it naturally lives in Fol_q. Foliations up to homotopy are essentially classified by the Pontryagin classes of their normal bundle (this is a consequence of the Bott-Haefliger theorems mentioned in [1]). We suspect then that the K-Theory constructed using the Quillen-Segal construction and the topological category $\Gamma_q(M)$ for

a manifold M might be closely related to Atiyah's ordinary K-Theory. What seems to be more interesting is probably the K-Theory arising from $Fol_q(M)$. This *was not included* in our list of K-Theories in [1], although it appears that it is probably the most appropriate one since we were using the GV-class.

For an *open* smooth n-manifold M one has a way to determine $Fol_q(M)$ by means of $\Gamma_q(M)$ using the Philips, Gromov, Haefliger (PGH) theorem as follows (see [6] for more details):

Let F be a codim-q foliation of M. Then the tangent bundle TM of M splits as $TM = F \oplus \nu(F)$ where $\nu(F)$ is the normal bundle of the foliation. Let $g_M : M \to BGL(n; \mathbf{R})$ denote the *Gauss map* that determines the tangent bundle TM of M. From the splitting of TM we deduce that the Gauss map g_M admits a homotopy lift $G_M : M \to BGL(q; \mathbf{R}) \times BGL((n-q); \mathbf{R})$. Since $\nu(F)$ is the normal bundle of the foliation, then G_M admits a second homotopy lift $\tilde{G_M} : M \to B\Gamma_q \times BGL((n-q); \mathbf{R})$. Then one has the following commutative diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\tilde{G_M}} & B\Gamma_q \times BGL((n-q); \mathbf{R}) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle B\nu \times id} \\
M & \xrightarrow{G_M} & BGL(q; \mathbf{R}) \times BGL((n-q); \mathbf{R}) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle p} \\
M & \xrightarrow{g_M} & BGL(n; \mathbf{R})
\end{array}
\tag{1}
$$

Then the (PGH) theorem says that there is a 1:1 correspondence between elements of $Fol_q(M)$ and the set of homotopy classes of homotopy lifts $\tilde{G_M}$ of g_M.

The map $B\nu : B\Gamma_q \to BGL(q; \mathbf{R})$ is the corresponding continuous map on the classifying spaces of the functor $\nu : \Gamma_q \to GL(q; \mathbf{R})$ which defines the normal bundle of the foliation just by considering the Jacobian of any local diffeomorphism (restricted to each leaf this is just the infinitesimal germinal holonomy of the leaf as we described above). (See also [1]).

Let us now pass on to the notion of *tautness* for foliations of arbitrary codim (> 1). There is exactly the same definition for geometric tautness using Riemannian metrics. Yet instead of topological tautness there is an analogous notion called *homological tautness*, which reduces to the well-known definition for the codim-1 case (see [3] p266). Sullivan's theorem then states that a codim-q foliation is geometrically taut iff it is homologically taut. An analogous characterisation using differentail forms also exists in this case, hence a codim-q foliation F on a closed smooth n-manifold M is taut iff there exists an $(n-q)$-form θ on M which is F-*closed* and transverse to the foliation. Transverse means it is non-singular when restricted to all leaves and the condition of being F-closed is a weaker condition than being closed; it means that $d\theta = 0$ whenever *at least* $(n-q)$ (i.e. same number as dim of the foliation) *of the vectors are tangent to F*.

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